

On the Variable-coefficient Burgers-Hlavaty Equation

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We study Hlavaty's generalization of the Burgers equation containing certain coefficient functions. We obtain a new auto-Bäcklund transformation and a family of exact analytical solutions along with the constraints on those coefficients.

Key words: Nonlinear Evolution Equations; Variable Coefficients; Bäcklund Transformation; Exact Solutions; Symbolic Computation.

In this paper we consider Hlavaty's variable-coefficient Painlevé-admissible extension of the Burgers equation [1, 2]

$$L(x, t) u_t + \frac{u^2 L_x(x, t)}{L(x, t)} + \frac{2 u L_x^2(x, t)}{L^2(x, t)} - 2 u u_x - \frac{u L_{xx}(x, t)}{L(x, t)} - u_{xx} + S(x, t) = 0, \quad (1)$$

where $L(x, t)$ and $S(x, t)$ are a couple of analytical functions. Recently, the prolongation structure with the related issues for (1) has been studied by Karasu [3].

We will investigate the interesting case of $L \neq 0$. Computerized symbolic computation will be used. Firstly we plan to use the Painlevé expansion [4, 5]

$$u(x, t) = \phi^{-J}(x, t) \sum_{j=0}^{\infty} u_j(x, t) \phi^j(x, t), \quad (2)$$

where J is a natural number and $\phi=0$ defines the singular manifold. Balancing powers of ϕ at the lowest orders requires that $J=1$. Next we truncate the expansion at the constant level terms, i.e.

$$u(x, t) = \phi^{-1}(x, t) \sum_{j=0}^1 u_j(x, t) \phi^j(x, t), \quad (3)$$

aiming to obtain a certain Bäcklund transformation and analytical solutions of (1), disregarding the integrability issue. Recent work in this direction is seen, e.g. in Ref. [6, 7].

When substituting (3) into (1), we make the coefficients of like powers of ϕ to vanish, so as to get

$$\phi^{-3}: u_0 = \phi_x \quad \text{with} \quad \phi_x \neq 0, \quad (4)$$

$$\phi^{-2}: \Theta = -L^2 \phi_t + 2 L u_1 \phi_x + L_x \phi_x + L \phi_{xx} = 0, \quad (5)$$

$$\phi^{-1}: \Theta_x L - 2 L_x \Theta \equiv 0, \quad (6)$$

$$\phi^0: u_1 \text{ needs to satisfy the original equation, i.e.}$$

$$L u_{1,t} + \frac{u_1^2 L_x}{L} + \frac{2 u_1 L_x^2}{L^2} - 2 u_1 u_{1,x} - \frac{u_1 L_{xx}}{L} - u_{1,xx} + S = 0. \quad (7)$$

The set of equations (3–5) and (7) constitutes an *auto*-Bäcklund transformation, since the set is solvable. Let us have some explicitly-solved sample solutions in the following analysis.

Into (5) we substitute a few trial expressions,

$$\phi(x, t) = v(x) t^2 + \beta(x) t + \lambda(x), \quad (8)$$

$$u_1(x, t) = \delta(x) t^2 + \psi(x) t + \sigma(x), \quad (9)$$

$$L(x, t) = \varepsilon(x) t^2 + \mu(x) t + \omega(x) \quad (\text{assumed for a variable coefficient}), \quad (10)$$

where $\sigma(x)$, $v(x)$, $\beta(x)$, $\delta(x)$, $\psi(x)$, $\varepsilon(x)$, $\mu(x)$, $\omega(x)$, and $\lambda(x)$ are all differentiable functions. Then, equating to zero the coefficients of like powers of t in (5) yields

$$t^6: \delta(x) = 0, \quad (11)$$

$$t^5: \text{the choice of } \varepsilon(x) = 0 \text{ with the understanding that } \mu(x) \neq 0, \quad (12)$$

$$t^4: v(x) = \text{constant} = v, \quad (13)$$

$$t^3: -v \mu(x) + \psi(x) \beta'(x) = 0 \rightarrow \text{for simplicity we choose } v(x) = 0 \text{ and } \beta(x) = \text{constant} = \beta \neq 0, \quad (14)$$

$$t^2: \psi(x) = \frac{\beta \mu(x)}{2 \lambda'(x)}, \quad \text{where } \lambda'(x) \neq 0, \quad (15)$$

$$t^1: \sigma(x) = -\frac{\mu'(x)}{2 \mu(x)} + \frac{\beta \omega(x) - \lambda''(x)}{2 \lambda'(x)}, \quad (16)$$

$$t^0: \omega(x) = \alpha \mu(x), \quad (17)$$

where α is a constant. This way, (5) has been satisfied, so that those trial expressions are allowed. Correspondingly, (7) implies that

$$S(x, t) = \sum_n S_n(x) t^n, \quad (18)$$

where $S_n(x)$'s are also differentiable. We substitute (18) into (7) and again equate to zero the coefficients of like powers of t , yielding (21) at the end of this paper.



Lastly, having considered (3) and (4) and put everything together, we obtain

$$u(x, t) = \frac{\beta t \mu(x)}{2 \lambda'(x)} + \frac{\lambda'(x)}{\beta t + \lambda(x)} - \frac{\mu'(x)}{2 \mu(x)} + \frac{\alpha \beta \mu(x) - \lambda''(x)}{2 \lambda'(x)} \quad (19)$$

with the constraints on the coefficient functions as

$$L(x, t) = (\alpha + t) \mu(x), \quad (20)$$

$$\begin{aligned} S(x, t) = & -\frac{\alpha \beta \mu(x)^2}{2 \lambda'(x)} - \frac{\beta t \mu(x)^2}{2 \lambda'(x)} + \frac{\alpha^2 \beta^2 \mu(x) \mu'(x)}{4 \lambda'(x)^2} + \frac{\alpha \beta^2 t \mu(x) \mu'(x)}{2 \lambda'(x)^2} + \frac{\beta^2 t^2 \mu(x) \mu'(x)}{4 \lambda'(x)^2} \\ & - \frac{\alpha \beta \mu'(x)^2}{2 \mu(x) \lambda'(x)} - \frac{\beta t \mu'(x)^2}{2 \mu(x) \lambda'(x)} - \frac{3 \mu'(x)^3}{4 \mu(x)^3} - \frac{\alpha^2 \beta^2 \mu(x)^2 \lambda''(x)}{2 \lambda'(x)^3} - \frac{\alpha \beta^2 t \mu(x)^2 \lambda''(x)}{\lambda'(x)^3} \\ & - \frac{\beta^2 t^2 \mu(x)^2 \lambda''(x)}{2 \lambda'(x)^3} - \frac{\alpha \beta^2 \mu'(x)^2 \lambda''(x)}{2 \lambda'(x)^2} - \frac{\beta t \mu'(x) \lambda''(x)}{2 \lambda'(x)^2} + \frac{2 \alpha \beta \mu(x) \lambda''(x)^2}{\lambda'(x)^3} \\ & + \frac{2 \beta t \mu(x) \lambda''(x)^2}{\lambda'(x)^3} - \frac{3 \mu'(x) \lambda''(x)^2}{4 \mu(x) \lambda'(x)^2} - \frac{3 \lambda''(x)^3}{2 \lambda'(x)^3} + \frac{\alpha \beta \mu''(x)}{2 \lambda'(x)} + \frac{\beta t \mu''(x)}{2 \lambda'(x)} \\ & + \frac{3 \mu'(x) \mu''(x)}{2 \mu(x)^2} - \frac{\alpha \beta \mu(x) \lambda^{(3)}(x)}{\lambda'(x)^2} - \frac{\beta t \mu(x) \lambda^{(3)}(x)}{\lambda'(x)^2} + \frac{\mu'(x) \lambda^{(3)}(x)}{2 \mu(x) \lambda'(x)} \\ & + \frac{2 \lambda''(x) \lambda^{(3)}(x)}{\lambda'(x)^2} - \frac{\mu^{(3)}(x)}{2 \mu(x)} - \frac{\lambda^{(4)}(x)}{2 \lambda'(x)} \end{aligned} \quad (21)$$

where the differentiable functions $\mu(x)$ and $\lambda(x)$, and the constants α and β all remain arbitrary, except that $\lambda'(x) \neq 0$, $\mu(x) \neq 0$ and $\beta \neq 0$.

In conclusion, for (1) we have obtained the new family of exact analytical solutions (19–21), as well as a new *auto*-Bäcklund transformation as stated above.

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